James boundaries and σ-fragmented selectors

B. Cascales

Universidad de Murcia

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The co-authors

- B. C and **M. Muñoz and J. Orihuela**, *James boundaries and σ-fragmented selectors*, Preprint. 2007. Available at http://misuma.um.es/beca

- B. C, **V. Fonf, J. Orihuela**, and **S. Troyanski**, *Boundaries in Asplund spaces*, Preprint 2007.

Boundaries: definitions

Throughout the lecture...

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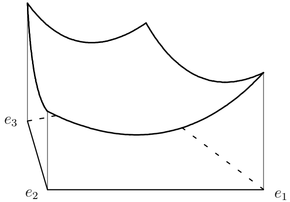

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Problem 1: The boundary problem (Godefroy)...extremal test

Let X Banach space, $B \subset B_{X^*}$ boundary and denote by $\tau_p(B)$ the topology defined on X by the pointwise convergence on B. Let H be a norm bounded and $\tau_p(B)$-compact subset of X.

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Problem 2: When is a boundary strong?

Let X Banach space, $B \subset B_{X^*}$ boundary.

When do we have $B_{X^*} = \overline{\mathrm{co} B}^{\| \ \|}$?

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Right $\Longrightarrow$ Left. Left is open in full generality. Right isn't always true.

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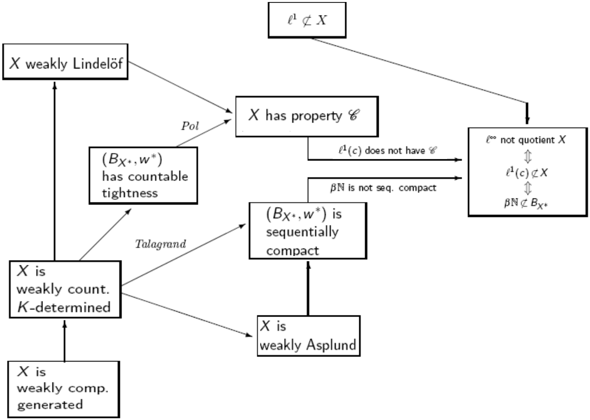

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4. We characterize Banach spaces X without copies of ℓ^1 via *boundaries* extending the results by Godefroy for the separable case.
5. For Asplund spaces we characterize boundaries for which $K = \overline{\text{co}\, B}^{\|\ \|}$. We extend in several different ways results by Namioka and Fonf.

Our first result: answer to a question by Plichko

Proposition, Muñoz-Orihuela-B.C.

Let X be a Banach space, B a boundary for B_{X^*}, $1 > \varepsilon \geq 0$ and $T \subset X^*$ such that $B \subset \bigcup_{t \in T} B(t, \varepsilon)$. If (T, w) is countably K-determined (resp. K-analytic) then:

(i) $X^* = \overline{\text{span}\, T}^{\|\ \|}$ and X^* is weakly countably K-determined (resp. weakly K-analytic).

(ii) Every boundary for B_{X^*} is strong. In particular $B_{X^*} = \overline{\text{co}(B)}^{\|\ \|}$.

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This answers a question by Plichko, extends Godefroy's result for separable boundary and improves Contreras-Payá and Fonf-Lindenstrauss result.

2nd result: characterization of Asplund spaces via selectors

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The following conditions are equivalent for a Banach space X:

(i) X is an Asplund space;

(ii) J has a Baire one selector;

(iii) J has a σ-fragmented selector;

(iv) for some $0 < \varepsilon < 1$, J has an ε-selector that sends norm separable subsets of X into norm separable subsets of X^*.

(v) there exists $0 < \varepsilon < 1$ such that (B_{X^*}, w^*) is ε-fragmented, *i.e.*, for every non-empty subset $C \subset B_{X^*}$ there exists some w*-open set V in B_{X^*} such that $C \cap V \neq \emptyset$ and $\| \, \| - \mathrm{diam}(C \cap V) < \varepsilon$.

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(iv) for some $0 < \varepsilon < 1$, J has an ε-selector that sends norm separable subsets of X into norm separable subsets of X^*.

(v) there exists $0 < \varepsilon < 1$ such that (B_{X^*}, w^*) is ε-fragmented, *i.e.*, for every non-empty subset $C \subset B_{X^*}$ there exists some w*-open set V in B_{X^*} such that $C \cap V \neq \emptyset$ and $\| \, \| - \mathrm{diam}(C \cap V) < \varepsilon$.

Duality mapping

If $(X, \| \, \|)$ is a Banach space the duality mapping $J : X \to 2^{B_{X^*}}$ is defined at each $x \in X$ by

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\}.$$

2nd result: characterization of Asplund spaces via selectors

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Notes:

- Borel measurable maps are σ-fragmented.
- The implication (ii)$\Rightarrow$(i) is proved in [JR02] with extra hypothesis which are justified with a wrong example.
- The equivalence with (v) is known when we write *for every* ε: a different proof has been given quite recently by Fabian-Montesinos-Zizler.

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(i) $\ell^1 \not\subset X$;

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Let X be an Asplund space, K a w*−compact convex subset of the dual space X^* and $B \subset K$ a boundary of K. Each one of the condition below implies that $K = \overline{\mathrm{co}\, B}^{\|\ \|}$:

(i) B is γ-closed.

(ii) B is w*-K-analytic.

Boundaries and the topology γ

The techniques now are *topological* techniques developed by Namioka-Orihuela-B. C and Namioka-B. C.

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Two open problems

1. The boundary problem in full generality (Godefroy).
2. Characterize strong boundaries out of the setting of Asplund spaces.

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